Majorizer Design for Non-Cartesian MRI with Sparsity-Promoting Regularization

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Introduction: Combinations of parallel MRI and compressed sensing have been proposed for reducing MRI scan times.[1,2] BARISTA[3] is a fast algorithm for parallel MRI and compressed sensing due to its consideration of coupling between the sensitivity maps and the spatial localization property of the wavelet transform, but it remains untested in the non-Cartesian setting. This abstract extends this approach to non-Cartesian trajectories by considering coupling between the frequency localization property of the wavelet transform and the density compensation inherent in non-Cartesian sampling patterns. The cost function associated with parallel MRI and compressed sensing has the form, $\Psi(x) = \frac{1}{2}\|y - Ax\|_2^2 + \beta\|Rx\|_1$, where A is a SENSE system matrix and R is an undecimated wavelet transform. $A = FS$ where F is a non-Cartesian Fourier operator and S is a matrix that involves the sensitivity maps. The BARISTA approach would upper bound $F'F$ with LI, where L is the maximum eigenvalue of $F'F$. This bound is loose in the non-Cartesian setting due to the high density of samples at low frequencies. To accelerate convergence, we bound $F'F$ more tightly with M_f, a circulant matrix that is related to the density compensation function of $F'F$. This has been explored previously, but previous works used a stochastic, moving-target cost function with orthogonal wavelets that had weaker convergence guarantees.[4] We build on advantages of previous approaches and derive a BARISTA-type algorithm with varying substeps for each frequency subband of the wavelet transform. The new algorithm is observed to accelerate convergence by a factor of 2-3 over standard FISTA[5] in numerical experiments.

Methods: The matrix, $F'F$, can be upper bounded with a circulant matrix, M_f, by first computing $D_f = \mathrm{diag}(QFe_0)$, where Q is a DFT matrix and e_0 is an impulse response. We then increase the minimum entry in D_f until $M_f = Q^{-1}D_fQ$ is a majorizer of $F'F$ as determined via power iteration.[6] We found empirically that this matrix is also a majorizer for $A'A$ when the sensitivity maps are calculated with a square root sum of squares normalization. Once M_f is calculated, we then need to determine D_R such that $D_R \succcurlyeq RM_f^{-1}R^T$. D_R is a matrix that governs step sizes for regularization.[3] We first note that $R^T = [R_1^T \dots R_B^T]$ for an undecimated wavelet transform with B subbands. For each subband, we have $R_b = Q^{-1}\Lambda_b Q$. Inserting this into $RM_f^{-1}R^T$ reveals that we can apply Geršgorin's Theorem to build $D_R = \mathrm{diag}(d_{R,b}I_b)$, where I_b is an identity matrix of the size corresponding to the bth subband and $d_{R,b} = \max_n(\sum_{p=1}^{B}|\lambda_{n,b}\lambda_{n,p}^*/d_{n,f}|)$, where $\lambda_{n,b}$ is the nth entry in Λ_b and $d_{n,f}$ is the nth entry in D_f. Within BARISTA, this gives the analysis denoising step of $q^{(j+1)} = P_{pM}(v^{(j)} - \beta^{-1}D_R^{-1}Rx^{(k,j+1)})$ where $x^{(k,j+1)} = c^{(k)} - \beta M_f^{-1}R^Tv^{(j)}$ and $c^{(k)} = x^{(k)} - M_f^{-1}A'(Ax^{(k)} - y)$. We compared the convergence speed of this new algorithm to that of FISTA[5] in a radial sampling scheme where 30 of the full 401 radial spokes with 8 coils were used for reconstruction. R was an undecimated Haar wavelet transform.

Results: We plot $\xi(k) = \frac{\|x^{(k)} - x^{(\infty)}\|}{\|x^{(\infty)}\|}$, the norm-residual to convergence, vs. time in all figures. $x^{(\infty)}$ was calculated by running many thousands of iterations. Figure 1 shows $x^{(\infty)}$ and $x^{(5)}$ (image after 5 iterations) for each algorithm in these numerical experiments. Figure 2 compares the convergence speed of the algorithms in the undecimated Haar wavelet case, showing the 2-3 factor increase in convergence speed.

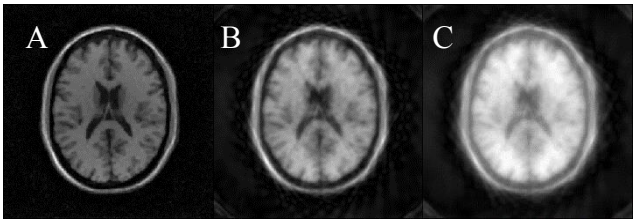

Figure 1: (A) The converged $x^{(\infty)}$ for the numerical experiments. (B) $x^{(5)}$ with the proposed method. (C) $x^{(5)}$ with FISTA, which is blurred without the proposed method.

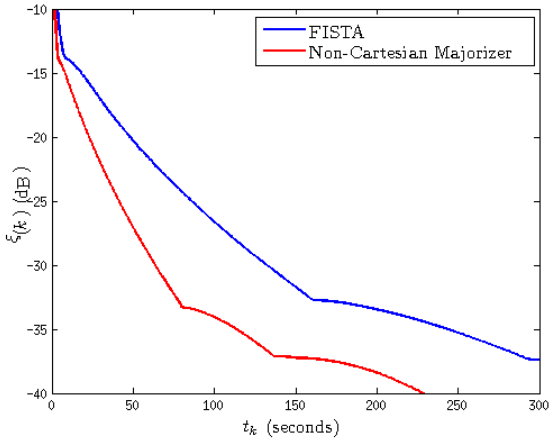

Figure 2: Convergence speed result comparing the proposed Non-Cartesian Majorizer to FISTA.

Discussion: We have demonstrated the utility of upper bounding non-Cartesian Hessian matrices in the SENSE parallel MRI setting with circulant majorizers. The circulant majorizers have frequency responses similar to that of the standard density compensation function. We further showed that this property can be exploited in the setting where the regularizing matrix, R, is an undecimated Haar wavelet transform. This gives subband-dependent stepsizes that can be computed without power iterations as soon as the circulant majorizer M_f is determined. In the future we plan to examine strategies for computing tighter circulant majorizers and guarantees that they upper bound $A'A$.

References: 1. Lustig et al., MRM 2007, 2. Ramani et al., IEEE-TMI 2011, 3. Muckley et al., IEEE-TMI 2015, 4. Guerquin-Kern et al. IEEE-TMI 2011, 6. Beck and Teboulle SIAM 2009, 6. Ramani and Fessler ISBI 2013.